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# NUMERICAL IMPLEMENTATION OF THREE-DIMENSIONAL FRICTIONAL CONTACT BY A LINEAR COMPLEMENTARITY PROBLEM

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The complementarity principle recently derived for a general three dimensional frictional contact is explicitly implemented as a linear complementarity problem (LCP). The inherent nonlinearity in the three dimensional friction condition has been treated by introducing a polyhedral law instead of elliptic law. The two-dimensional formulation previously derived is shown to be a special case of this three-dimensional formaulation in LCP.

**Key Words:** Complementarity Problem, Frictional Contact, Polyhedral Law of Friction, Orthotropic Friction Law, General Contact Condition

## 1. INTRODUCTION

Contact problems have long been a topic of intensive study since the Hertzian theory developed in 1881, mostly through analytical approach (Johnson, 1985) before 1970's and later through computational studies. (Conry and Seireg, 1971, Chan and Tuba, 1971, Francavilla and Zienkiewicz, 1975, Chand, Haug and Rim, 1976, Tsuta and Yamaji, 1973, Herrmann, 1978, Pian and Kubomura, 1981, Rahman et al., 1984, Bathe and Chaudhary, 1985, Mehlhorn et al, 1985, Chandrasekaran et al., 1987 and so on).

In solid mechanics, contact problem is one of the most difficult topics, being highly nonlinear: The geometric nonlinearity comes from the unknown kinematic boundary condition, that is, the unknown contact area, while the material nonlinearity from the frictional property.

For the contact without friction, general theories are available and well applicabe for efficient numerical treatments (Haug and Kwak, 1987, Park and Kwak, 1986, Lee and Kwak, 1984, Joo and Kwak, 1986, Lee and Kwak, 1989). The main theory is that an equivalent optimization exists; the minimization of the potential energy under some kinematic constraints (Panagiotopoulos, 1985).

In the friction case, the nonlinearity also involves irreversible and dissipative nature and hence it is loading-path dependent as is well known (Klarbring, 1986, Kwak and Lee, 1988, Martins and Oden, 1983). Therefore, few theories have been available for this difficult but important frictional contact problems. Even the very few theories appearing in the literature have still significant restrictions, or can deal with only a limited scope of problems.

The variational inequality formulation (Duvaut and Lions, 1976), for example, is limited to the case where either the normal contact force is assumed known or the contact area constant. The subsequent formulations by other authors (Oden and Pires, 1983, Kalker, 1988, Klarbring, 1988) have similar restictions.

Recently the author (Kwak, 1989) has obtained a complementarity principle which is directly derived from contact compatibility and friction conditions for general three-dimensional orthortropic friction law of the Coulomb type. Up to now, no principle has been available in the literature, for a three-dimensional friction problem. This derivation is completely new. In a previous paper by the author and Lee (1988), a formulation confined only to two-dimensional problems is derived and a detailed implementation by the boundary element method shown. An extension of the same concept to three dimensional problems has not been obvious at all. As is shown here, however, the two dimensional formulation is a special case of the three dimensional formulation with the polyhedral law of friction.

In this paper, the nonlinear complementarity relation (Kwak, 1989) is implemented as an LCP by the polyhedral law of friction substituting original elliptic law. The resulting formulation is neat and very suitable for efficient numerical treatment. It is general, because it covers full three-dimensional contacts with an orthotropic friction law. Also, there is no difference whether a multi-body contact or a contact against a rigid body is considered. Another special feature of the method is the inclusion of rigid-body degree-of-freedoms, which is not usually considered in the literature although very important in practice.

Since an incremental approach is inevitable for the path-dependency of friction effect, usual nonlinearities such as large deformation and elasto-plasticity can be easily incorporated. The property of the proposed formulation in terms of uniqueness for an incremental step will be studied and presented elsewhere with numerical testings.

## 2. PROBLEM FORMULATION

For the purpose of reference and continuity, the complementarity principle derived in (Kwak, 1989) is repeated

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briefly. It is described for a large deformation, three-dimensional contact problem with an orthortropic friction law of the Coulomb type.

In the incremental approach, the configuration and state up to time t (or a time-like parameter) are known and those at time $t+\Delta t$ are to be sought. The updated Lagrangian approach is taken for illustration. In this method, all the static and kinematic state variables are referred to the configuration at time $t$. Following the usual notation in texts, let the coordinates of a generic point in a body be denoted as $a_i$ and $x_i$ at time $t$ and $t+\Delta t$, respectively. The displacement increment during $\Delta t$ is then,

$$u_i = x_i - a_i \tag{1}$$

The energy-conjugate strain and stress measure referred to the configuration at $t$ are the Lagrangian strain tensor $\varepsilon_{ij}$ and the second Piola-Kirchhoff stress tensor ${}^{t+\Delta t}S_{ij}$. They are described by,

$$\varepsilon_{ij} = e_{ij} + \eta_{ij} \tag{2}$$

where,

$$e_{ij} = (u_{i,j} + u_{j,i})/2 \text{ and } \eta_{ij} = u_{k,i} u_{k,j}/2 \tag{3}$$

The derivative $(\ )_{,j}$ means differentiation with respect to the coordinate $a_j$. The second Piola-Kirchhoff stress tensor has the following relationship with the Cauchy stress tensor ${}^{t+\Delta t}\sigma_{ij}$, which represent the stress at $t+\Delta t$ referred to the configuration at $t+\Delta t$.
That is,

$${}^{t+\Delta t}\sigma_{ij} = |J| x_{i,r}\ {}^{t+\Delta t}S_{rs}\ x_{j,s} \tag{4}$$

where $|J|$ is the determinant of the Jacobian of the coordinate transformation. The second Piola-Kirchhoff stress is decomposed as follows,

$${}^{t+\Delta t}S_{ij} = {}^{t}\sigma_{ij} + S_{ij} \tag{5}$$

where $S_{ij}$ denotes the second Piola-Kirchhoff stress increment tensor. It is noted that in the following the quantities without time specifications by a left superscript denote corresponding increments from $t$ to $t+\Delta t$.

The derivation is described for a two-body contact without loss of generality. Body 2 is assumed restrained against any

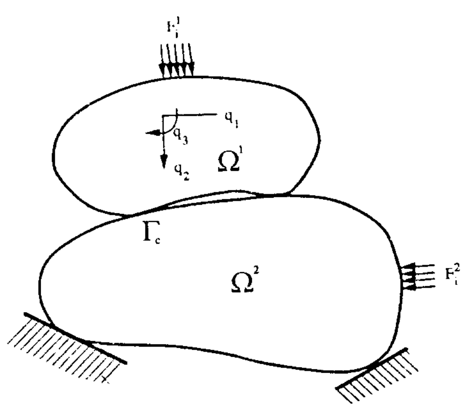


**Fig. 1** Two bodies in contact

rigid body displacements, while body 1 is allowed for rigid body displacement, denoted by $q_i$, suitably defined with respect to a point or a set of points, such that the displacement field of body 1 can be described by the sum of $q_i$ and the relative displacement with respect to the reference point or set(Chand, Haug and Rim, 1976, and Haug and Kwak, 1978). This situation is completely general and is shown in Fig 1. An extension to a multi-body contact requires only some additional book-keeping.

For the purpose of description, let the region occupied by the bodies be $\Omega^1$ and $\Omega^2$ and their boundaries by $\Gamma^1$ and $\Gamma^2$, respectively. Each boundary is composed of thress disjoint parts; $\Gamma_u$ where displacement is prescribed, $\Gamma_f$ where traction conditions are given and $\Gamma_c$ which is the so-called potential contact region taken to be sufficiently large to cover the real contact area after deformation.

The normal contact stress and tangential contact stress are denoted by $S_n$ and $S_t$. For body $k$, $k=1$ or 2;

$${}^{t}S_n{}^{k} = {}^{t}\sigma_{ij}{}^{k}\ {}^{t}n_i{}^{k}\ {}^{t}n_j{}^{k} \tag{6}$$
$${}^{t}S_{ti}{}^{k} = {}^{t}\sigma_{ij}{}^{k}\ {}^{t}n_j{}^{k} - {}^{t}S_n{}^{k}\ {}^{t}n_i{}^{k} \tag{7}$$

where $n$ and $t$ denote the unit normal and tangential vector, respectively. Let $P$ denote contact pressure taken to be positive by tradition. Then,

$${}^{t}P = -{}^{t}S_n{}^{1} = -{}^{t}S_n{}^{2} \tag{8}$$

The relations to be satisfied in an incremental step may now be described as follows.

(1) Global Equilibrium

For body 1, where a rigid body motion is allowed, all the external forces and the contact forces should be in equilibrium. For configuration at $t+\Delta t$, from the principle of virtual displacements (Haug and Kwak, 1978),

$$\int_{{}^{t+\Delta t}\Gamma_f^1} {}^{t+\Delta t}F_i \beta_{ij}\ {}^{t+\Delta t}d\Gamma + \int_{{}^{t+\Delta t}\Gamma_c^1} {}^{t+\Delta t}S_i \alpha_{ij}\ {}^{t+\Delta t}d\Gamma = 0 \tag{9}$$

where the coefficient matrices $\alpha_{ij}$ and $\beta_{ij}$ represent the rigid body displacements of points of ${}^{t+\Delta t}\Gamma_c$ and ${}^{t+\Delta t}\Gamma_f$ in the direction of $F_i$ and $S_i$ due to a unit displacement in the $j$-th rigid body degree-of-freedom, respectively. The notation $F_i$ and $S_i$ denote traction vectors corresponding to the external and contact force, respectively. If there is a body force field $f_i$, an integral over the domain with a kinematic matrixd similar to $\beta_{ij}$ or $\alpha_{ij}$ can be added. For notational simplicity, however, it will be suppressed throughout the paper.

(2) Internal Equilibrium of Each Body

The local equilibrium equations are expressed in terms of the second Piola-Kirchhoff stress tensor at configuration $t+\Delta t$ as follows (Joo and Kwak, 1986),

$$({}^{t+\Delta t}S_{jk} x_{i,k})_{,j} = 0 \text{ in } \Omega^k \tag{10}$$

(3) Strain-Displacement Relationship This is already given by Eqs. (2, 3)

(4) Stress-Strain Relationship for an Incremental Step

$$S_{ij} = C_{ijrs} \varepsilon_{rs} \tag{11}$$

where $C_{ijrs}$ is the constitutive coefficients at current time $t$.

(5) Boundary Conditions for Each Body

Displacement conditions are

$$u_i = U_i \text{ on } \Gamma_u^k \tag{12}$$

where $U_i$ is the given displacement increment. The traction boundary conditions are

$$^{t+\Delta t}\sigma_{ij}\ ^{t+\Delta t}n_j = {}^{t+\Delta t}F_i \text{ on } \Gamma_f^k \tag{13}$$

(6) Compatibility Condition

No material particle will penetrate into the surface of the opposing body. Let the opposing surfaces in the potential contact region at configuration $t$ be denoted by smooth functions,

$$g^1(a_i^1) = 0 \text{ and } g^2(a_i^2) = 0 \tag{14}$$

Then the impenetration condition says that all the particles $a_i^2$ should be outside body 1 at configuration $t+\Delta t$. This can be expressed as; for any point $a_i^1$, (Fig. 2)

$$g^1(a_i^2 + u_i^2(a_i^2) - \hat{u}_i^1(a_i^1)) \geq 0 \tag{15}$$

for all $a_i^2$ satisfying $g^2(a_i^2) = 0$ on $\Gamma_c^2$, where

$$\hat{u}_i^1(a_i^1) = u_i^1(a_i^1) + \alpha_{ij} q_j \tag{16}$$

By defining,

$$D_n(a_i^1) = \min\ g^1(a_i^2 + u_i^2(a_i^2) - \hat{u}_i^1(a_i^1)) \tag{17}$$

where the value $D_n(a_i^1)$, to be called as a gap function, essentially denotes the distance from the pariticle $a_i^1$ to the surface $g^2(a_i^2) = 0$ after deformation, the above condition can be simply expressed as

$$D_n(a_i^1) \geq 0 \text{ for all } a_i^1 \text{ such that } g^1(a_i^1) = 0 \tag{18}$$

The compatibility condition can now be described by

$$^{t+\Delta t}P D_n(a_i^1) = 0 \text{ for all } a_i^1 \text{ on } \Gamma_c^1 \tag{19}$$

This states that if there is nonzero pressure, than the gap must be zero, and vice versa, It is noted that, instead of finding $a_i^2$ exactly, an approximate contacting pair obtained from the configuration $t$ is often used.

(7) Friction Conditions

Let the principal orthotropic axes on the tangent plane at a point on $\Gamma_c^1$ be denoted locally by $T_a$ and $T_b$ with the corresponding coefficients of friction, $\mu_a$ and $\mu_b$ The tangential traction components and the relative tangential displacement components along the axes are $S_{Ta}$ and $S_{Tb}$, and $D_{Ta}$ and $D_{Tb}$, respectively. The form of the static friction law is the following.

$$\left[\left(\frac{^{t+\Delta t}S_{Ta}}{\mu_a}\right)^2 + \left(\frac{^{t+\Delta t}S_{Tb}}{\mu_b}\right)^2\right]^{1/2} \leq {}^{t+\Delta t}P \tag{20}$$

where $^{t+\Delta t}P$ is the contact pressure which is nonnegative by convention.

If strict inequality holds, there is no slip. Otherwise, a relative motion is imminent. In this latter case, it is known (Panagiotopoulos, 1985) that there exists a nonnegative $\lambda$ such that

$$D_{Ta} = -\lambda \frac{^{t+\Delta t}S_{Ta}}{\mu_a^2} \text{ and } D_{Tb} = -\lambda \frac{^{t+\Delta t}S_{Tb}}{\mu_b^2} \tag{21}$$

The relative displacements are defined from the tangential components of the vector$(u^2 - u^1)$ when body 1 is under focus For description, it is convenient to introduce local coordinate systems, such as $(T_a, T_b, n)^1$ and $(T_a, T_b, n)^2$ in Fig. 3. It is noted that axes $T_b^1$ and $T_b^2$ are in the same direction while the others are opposite each other. With respect to these coordinate systems, all the tangential components are defined. From action-reaction principle,

$$S_{Ta} = S_{Ta}^1 = S_{Ta}^2,\quad S_{Tb} = S_{Tb}^1 = -S_{Tb}^2,\quad S_n = S_n^1 = S_n^2$$
$$D_{Ta} = D_{Ta}^1 = D_{Ta}^2,\quad D_{Tb} = D_{Tb}^1 = -D_{Tb}^2 \text{ and so on.} \tag{22}$$

Here and in the following, variables without superscripts on the right are referred to body 1 and its local coordinates.

For a two-dimensional case, the Coulomb law says,

$$-\mu^{t+\Delta t}P \leq {}^{t+\Delta t}S_T \leq \mu^{t+\Delta t}P \tag{23}$$

When strict inequalities hold for both the equations, there is no relative motion; $D_T = 0$. Otherwise, slip is possible; $D_T = -\lambda S_T$ with $\lambda \geq 0$.

It is natural to transform the above statements directly to a so called complementarity problem form (Cottle et al., 1980). Some complementarity propety of the friction law has been recognized and utilized in the literature (Kwak and Lee, 1988, Klarbring, 1988), although limited in scope. The formulation proposed here is very general. First introduce a nonnegative real number $^{t+\Delta t}P^*$ and $^{t+\Delta t}\theta_f$ such that

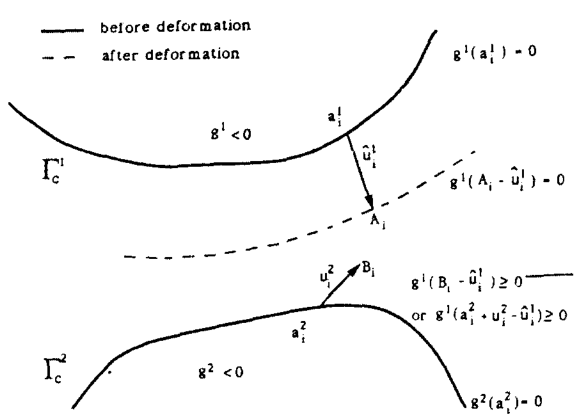


**Fig. 2** General impenetration condition

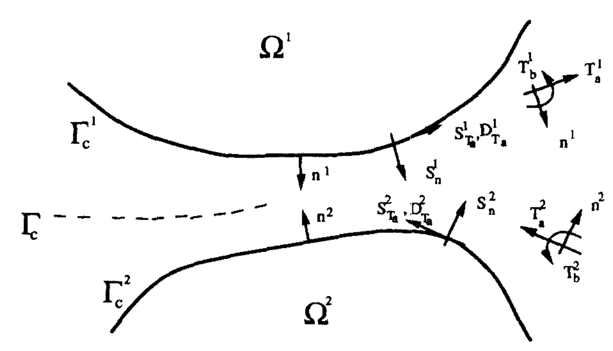


**Fig. 3** Principal orthotropic axes and force and relative displacement components

$$\frac{{}^{t+\Delta t}S_{Ta}}{\mu_a} = {}^{t+\Delta t}P^* \cos^{t+\Delta t}\theta_f$$
$$\frac{{}^{t+\Delta t}S_{Tb}}{\mu_b} = {}^{t+\Delta t}P^* \sin^{t+\Delta t}\theta_f \tag{24}$$

Also introduce $D_T$ and $\theta_u$ such that

$$D_{Ta} = D_T \cos\theta_u \quad \text{and} \quad D_{Tb} = D_T \sin\theta_u \tag{25}$$

Where $D_T$ denotes the magnitude of slip or the relative displacement between a pair of contacting points. Then the friction condition ,(20) can be expressed as

$$^{t+\Delta t}P^* + T = {}^{t+\Delta t}P \tag{26}$$

with the complementarity rule,

$$T \ D_T = 0, \tag{27}$$

where $T \geq 0$ is a slack variable, and $D_T \geq 0$. For the two-dimensional case, a similar complementarity condition has already been developed in Kwak and Lee(1988), but it is not directly derivable from the above three-dimensional statement. The connection between the two will only be clear after introduction of the polyhedral law of friction approximating the orginal friction law, Eq. (20). Another relation is dervied from Eq. (21) using Eqs. (24, 25) as follows,

$$\tan\theta_u = \frac{\mu_a}{\mu_b} \tan^{t+\Delta t}\theta_f \tag{28}$$

# 3. NONLINEAR COMPLEMENTARITY SYSTEM

The governing equations derived may be interpreted as follows. Eqs. (2, 3, 10, 11) with the boundary conditions (12) and (13) constitute the usual set of governing equations for deformation mechanics. They essentially provide a relationship between the contact traction vector and displacement vector on the potential contact region $\Gamma_c$. Conceptually, this relationship can be expressed as follows,

$$u_c^k = u_c^k({}^{t+\Delta t}S_c^k \,;\, {}^{t+\Delta t}F^k), \quad k=1, 2 \tag{29}$$

where the subscript $c$ refers to the potential contact region $\Gamma_c^k$ and ${}^{t+\Delta t}S_c^k = {}^tS_c^k + S_c^k$ denotes a traction vector with both normal and tangential components, ${}^{t+\Delta t}S_n$, ${}^{t+\Delta t}S_{Ta}$ and ${}^{t+\Delta t}S_{Tb}$. It is noted that an implicit relationship, say, from the principle of minimum energy is also good for the following development.

The global equilibrium Eq.(9) can be expressed symbolically,

$$E({}^tS_c + S_c \,;\, {}^tF + F) = 0 \tag{30}$$

Also, if Eq. (29) is substituted into (16) and (17), the gap function can be expressed as,

$$D_n({}^ta_i^1) = G^1({}^tS_c + S_c, \ q^+, \ q^- \,;\, {}^ta_i^2) \tag{31}$$

where $a_i^1$ and $a_i^2$ are mating contact points predetermined from Eq. (17) at configuration t, and the rigid body displacement is expressed as a difference of two nonnegative numbers, $q_i = q_i^+ - q_i^-$, where $q_i^+$and $q_i^-$ are nonnegative. This will be denoted as,

$$q = q^+ - q^- \tag{32}$$

where $q^+ \geq 0$ and $q^- \geq 0$.
Then the condition in Eq. (19) becomes

$$({}^tP + P) D_n = 0 \tag{33}$$
and
$$ {}^tP + P \geq 0 \text{ and } D_n \geq 0 \tag{34}$$

To get a consistent formulation, the following manipulation is introduced for Eq. (30). Since an equality relation can be equivalently stated by two inequalities, Eq. (30) can be substituted by

$$E({}^tS_c + S_c \,;\, {}^tF + F) + V^- = 0 \tag{35}$$

and

$$E({}^tS_c + Sc \,;\, {}^tF + F) - V^+ = 0 \tag{36}$$

where $V^+$ and $V^-$ are slack variables for the two inequalities. Then the following complementarity relations can be stated,

$$V^+ q^+ = 0 \text{ and } V^- q^- = 0 \tag{37}$$

where

$$V^+ \geq 0, \ q^+ \geq 0, \ V^- \geq 0, \ q^- \geq 0 \tag{38}$$

This follows since actually $V^+ = V^- = 0$. It is noted, however, that by adding Eqs. (35, 36) $V = V^+ - V^-$ is shown to have the meaning of a constrained force, which is zero when $q$ is allowed to take any value.

The friction conditions, Eqs .(24, 25, 26) when applied for the incremental analysis under consideration can be stated as follows.

$$({}^tP^* + P^*) + T = {}^tP + P \tag{39}$$

and

$$TD_T = 0, \tag{27}$$

where

$$\frac{{}^tS_{Ta} + S_{Ta}}{\mu_a} = ({}^tP^* + P^*) \cos({}^t\theta_f + \theta_f) \tag{40a}$$
$$\frac{{}^tS_{Tb} + S_{Tb}}{\mu_b} = ({}^tP^* + P^*) \sin({}^t\theta_f + \theta_f) \tag{40b}$$

and

$$D_{Ta} = D_T \cos\theta_u \text{ and } D_{Tb} = D_T \sin\theta_u \tag{25}$$

Now since $D_{Ta}$ and $D_{Tb}$ are funcitons of $u_c^1$ and $u_c^2$, they can be expressed in terms of $S_{Ta}$, $S_{Tb}$ and $S_n$ using Eq(29).

$$D_{Ta} = D_{Ta}(S_{Ta}, \ S_{Tb}, \ S_n) \tag{41a}$$
$$D_{Tb} = D_{Tb}(S_{Ta}, \ S_{Tb}, \ S_n) \tag{41 b}$$

Theoretically it is possible to solve for the free variables, $S_{Ta}$, $S_{Tb}$, $D_{Ta}$, $D_{Tb}$, $\theta_f$ and $\theta_u$ in terms of $T$, $P(=-S_n)$ and $D_T$ from the above 6 equations, when applied for each point on $\Gamma_c$. By substituting these solutions into Eqs. (28, 31, 35, 36) a formal complementarity problem is finally obtained with the complementarity relations (27), (33) and (37). The unknown variables are identified pairwise as,

$$[V^+,\ q^+;\ V^-,\ q^-;\ \{P,\ D_n,;\ T,\ D_t\}] \tag{42}$$

This is, however, recognized as a nonlinear complementarity system, even with linearly elastic material, because of Eqs (25, 40a,b) For a two dimensional case, a linear complementarity problem has been possible. This shows that the nature of the three dimensional frictional problem is essentially different from that of a two dimensional one, although a natural connection exists as will be shown in the following. It is also observed that it is not amenable to use a direct linearization method since the amounts $\theta_f$ and $\theta_u$ are not necessarily small, because a sudden change in the direction can occur, especially when a slip occurs from a sticking status.

# 4. TREATMENT BY LINEAR COMPLEMENTARITY PROBLEM FOR NUMERICAL IMPLEMENTATION

## 4.1 linearized Geometric Compatibility

The gap function, (17) for a mating contact pair, can be expanded to obtain the linearized form as follows,

$$D_n = g^1(a_i^2) + (u_k^2 - u_k^1)\dot{g}_k^1 - a_{kj}q_j\dot{g}_k^1 \tag{43}$$

where

$$\dot{g}_k^1 = \frac{\partial g^1}{\partial a_k^1}\Big|_{a_k^1 = a_k^2}$$

It is seen that by dividing Eq.(43) by $|\dot{g}_k|$, one obtains a measure of distance. The first term in (43) denotes the initial gap at configuration $t$. The other two terms represent the normal components of the relative and rigid body displacement, respectively. A similar formula has also been derived directly from the mechanics and used extensively in the literature (Chand, Haug and Rim, 1976, Haug and Kwak, 1978)

## 4.2 Approximate Frictional Conditions for Numerical Implementation

It has been indicated that a formal linearization may be made as a Newton type approach. This, however, turns out to be complicated and not numerically efficient. Therefore, in the treatment, the original elliptic cone of acceptable traction forces defined in Eq. (20) is replaced by an approximate polyhedral cone. One such replacement appears in (Klarbring, 1988) in an example for an approximate isotropic law.

Let $C_E$ and $C_P(e)$ be the original elliptic cone, Eq. (20) and the polyhedral cone, respectively, as follows;

$$C_E = \left\{(S_{Ta},\ S_{Tb},\ P) : \left[\left(\frac{{}^{t+\Delta t}S_{Ta}}{\mu_a}\right)^2 + \left(\frac{{}^{t+\Delta t}S_{Tb}}{\mu_b}\right)^2\right]^{1/2} \le {}^{t+\Delta t}P\right\} \tag{44}$$

and

$$C_P(e) = \left\{(S_{Ta},\ S_{Tb},\ P) : \left|\frac{{}^{t+\Delta t}S^T{}_{Ta}}{\mu_a}\cos\alpha_i + \frac{{}^{t+\Delta t}S_{Tb}}{\mu_b}\sin\alpha_i\right| \le {}^{t+\Delta t}P,\ i=1,\cdots e\right\} \tag{45}$$

where $\alpha_i$ is a directional angle in ($S_{Ta}$. $S_{Tb}$) plane and $e$ denotes the number of pairs of opposing faces of the polyhedral cone. It is noted that the term on the left hand side of inequality sign in (45) denotes the component of tangential force along the direction $\alpha_i$ The cross sectional shapes of the cones, $C_E$, $C_p(2)$ and $C_p(4)$ for $P=1$ are shown in Fig. 4. The cone $C_p(1)$ with $\alpha_1=0$ corresponds to a one dimensional case. The cone $C_P(2)$ may be termed as a rectangle law of friction due to its shape. This may be interpreted as the result of direct application of the one-dimensional law along each principal axis as follows;

$$|{}^{t+\Delta t}S_{Ta}| \le \mu_a{}^{t+\Delta t}P$$
$$|{}^{t+\Delta t}S_{Tb}| \le {}^{t+\Delta t}P \tag{46}$$

The other case with arbitrary $e$ is called as a polyhedral law of friction.

To arrive at a complementarity rule for the polyhedral law, introduce slack varibles $T_i^-$ and $T_i^+$ which are nonnegative such that

$$\frac{{}^{t+\Delta t}S_{Ta}}{\mu_a}\cos\alpha_i + \frac{{}^{t+\Delta t}S_{Tb}}{\mu_b}\sin\alpha_i + \frac{T_i^-}{\mu^*} = {}^{t+\Delta t}P \tag{47a}$$

$$\frac{{}^{t+\Delta t}S_{Ta}}{\mu_a}\cos\alpha_i + \frac{{}^{t+\Delta t}S_{Tb}}{\mu_a}\sin\alpha_i - \frac{T_i^+}{\mu^*} = -{}^{t+\Delta t}P,\ i=1,\cdots,e \tag{47b}$$

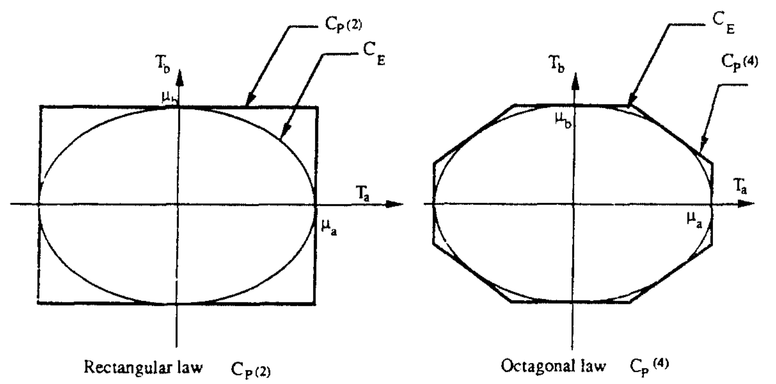


**Fig. 4** Polyhedral law of friction as compared to elliptical law

where

$$\mu^* = \sqrt{(\mu_a^2 + \mu_b^2)/2} \tag{48}$$

This term is introduced for later use. Define the relative displacement $D_{Ti}^+$ and $D_{Ti}^-$ for the corresponding directions $\beta_i$, as follows,

$$D_{Ta}\cos\beta_i + D_{Tb}\sin\beta_i = D_{ti}^+ - D_{ti}^-, \quad i = 1 \cdots\cdots e \tag{49}$$

where

$$D_{Ti}^+ \geq 0, \quad D_{Ti}^- \geq 0, \tag{50}$$

and

$$\tan\beta_i = \frac{\mu_a}{\mu_b}\tan\alpha_i \tag{51}$$

This last relation comes form Eq. (28). If the component of a tangential force vector, $(S_{Ta}, \quad S_{Tb})$, computed along the direction $\alpha_i$, falls on the boundary of the polyhedral cone, a slip in the opposite direction to the force will occur along $\beta_i$. That is, when $D_{Ti}^- > 0$, $T_i^-$ must be zero and when $T_i^- > 0$, there can not be a slip in the negative direction. This condition can be stated as

$$T_i^- D_{ti}^- = 0, \ i = 1 \cdots, \ e \tag{52a}$$

Similarly

$$T_i^+ D_{ti}^+ = 0, \ i = 1, \cdots e \tag{52b}$$

where

$$T_i^+ \geq 0, \ D_{Ti}^+ \geq 0, \ T_i^- \geq 0, \ D_{Ti}^- \geq 0.$$

Upon substitution into Eq. (47a, b), a set of equations are obtained. From the resulting equations, solve for $S_{Ta}$, and $S_{Tb}$ in terms of $P(= -S_n)$ and $T_i^-$ and $T_i^+$. These solutions are then substituted into Eqs. (30, 43, 49) resulting in the final linearized complementarity problem, with the primary unknowns as follows;

$$\begin{gathered}[V^+, \ q^+ ; V^-, \ q^- ; \\ \{P, \ D_n ; (T_i^+, \ D_{Ti}^+ ; T_i^-, \ D_{Ti}^- ; i = 1, \cdots e)\}]\end{gathered} \tag{53}$$

and the complementarity, (33), (37) and (52a, b). It is noted that when $C_p(1)$, corresponding to a two dimensional case is used, the formulation described in Kwak and Lee (1988) is obtained, as a special case of the three dimensional problem. For numerical implementation, it is convenient to impose Eqs. (47a, b, 49) for two angles with 90°apart at a time. This corresponds to imposing a rectangular law of friction one after another.

After an incremental step for $\Delta t$ is processed, the configuration at $t + \Delta t$ is taken as a new reference configuration for the next increment. Correspondingly, the coordinates and the state variables must all be updated, such that $a_i^{\text{new}} = a_i + u_i$, $q_j^{\text{new}} = q_j^{\text{old}} + q^j$ and so on. With the configuration change, the orthotropy direction for friction must also be updated.

# 5. NUMERICAL IMPLEMENTATION SCHEME BY LCP USING POLYHEDRAL LAW OF FRICTION

For the following development, it is considered that the contacting pair of points mentioned in the previous section has been approximately obtained already. Corresponding to a nodal point $a_i^1$, the opposing potential contact point $a_i^2$ will not fall on a nodal point in general. A suitable interpolation can always be used to find the values of necessary state variables in terms of those at neighboring nodes.

As methods of discretization, both FEM and BEM can equally be used to find the relationship (29). As already noted earlier, the various displacement components along the normal and tangential directions on the common potential contact surface can be expressed as follows, referring to Fig.3 and Eqs. (8, 22)

$$\begin{Bmatrix} u_n \\ u_a \\ u_b \end{Bmatrix} = \begin{Bmatrix} u_n^1 + u_n^2 \\ u_a^1 + u_a^2 \\ u_b^1 - u_b^2 \end{Bmatrix} = \begin{bmatrix} -Q_{nn} & Q_{na} & Q_{nb} \\ -Q_{an} & Q_{aa} & Q_{ab} \\ -Q_{bn} & Q_{ba} & Q_{bb} \end{bmatrix} \begin{Bmatrix} P \\ S_a \\ S_b \end{Bmatrix} + \begin{Bmatrix} R_n \\ R_a \\ R_b \end{Bmatrix} F \tag{54}$$

where the minus signs in the third terms come from the replacement of $-P$ in place of $S_n$. It is noted that the original interface stiffness corresponding to $(S_n, \ S_a, \ S_b)$ is symmetric positive definite for a stable equilibrium. For notational simplicity in the numerical scheme, the subscript $T$ used up to now to refer to the tangential direction will be omitted in the following.

As already noted, it is convenient to impose the polyhedral cone condition pairwise. Therefore, consider angles $\alpha_1$ and $\alpha_2$ first. The description for angles $\alpha_3$ and $\alpha_4$ will be exactly the same. For notational simplicity, let

$$\begin{aligned} \mu_a^* &= \mu_a/\mu^* (\sin\alpha_2\cos\alpha_1 - \sin\alpha_1\cos\alpha_2) \\ \mu_b^* &= \mu_b/\mu^* (\cos\alpha_2\sin\alpha_1 - \cos\alpha_1\sin\alpha_2) \end{aligned} \tag{55}$$

Then, by letting

$$\frac{\bar{S}_a}{\mu_a}\cos\alpha_i + \frac{\bar{S}_b}{\mu_b}\sin\alpha_i \equiv \frac{\bar{S}_i}{\mu^*} \tag{56}$$

from Eq. (47a, b)

$$\begin{aligned} \bar{S}_i + T_i^- &= \mu^* \bar{P} \\ \bar{S}_i - T_i^+ &= -\mu^* \bar{P} \end{aligned} \tag{57}$$

where ($\bar{\ }$) denotes values at configuration $t + \Delta t$. Solving for $S_a$ and $Sb$ from these equations,

$$\begin{pmatrix} \bar{S}_a \\ \bar{S}_b \end{pmatrix} = \frac{1}{2} \begin{bmatrix} -M_a^2 & M_a^1 & M_a^2 & -M_a^1 \\ -M_b^2 & M_b^1 & M_b^2 & -M_b^1 \end{bmatrix} T \tag{58}$$

where $T = [T_1^-, \quad T_2^-, \quad T_1^+, \quad T_2^+]^T$, with $T_1^-$ itself being the vector of $T_1^-$ for each nodal point, and $M_a^1$, $M_a^2$, $M_b^1$ and $M_b^2$ are diagonal matrices with $(\mu_a{}^*\sin\alpha_1)_j$, $(\mu_a{}^*\sin\alpha_2)_j$, $(\mu_b{}^*\cos\alpha_1)_j$ and $(\mu_b{}^*\cos\alpha_2)_j$, respectively. The subscript $j$ refers to the nodal numbering on $\Gamma_c$.

Now the gap and slip can be expressed as

$$\begin{Bmatrix} D_n \\ D_a \\ D_b \end{Bmatrix} = \begin{Bmatrix} -u_n - A_n q + \delta \\ u_a + A_a q \\ u_b + A_b q \end{Bmatrix} \tag{59}$$

where $\delta$ corresponding to the $g^1(a_i^2)/|\dot{g}^1|$ in Eq.(43) denotes the gap at current configuration $t$ between the contacting pair and $A=[a_{ij}]$ is appropriately partitioned as $A=[A_n,\ A_a,\ A_b]$ corresponding to $(n,\ a,\ b)$ directions at each nodal point of $\Gamma_c^1$. By substituting expression (58) into (54) and their results into (59),

$$\begin{Bmatrix} D_n \\ D_a \\ D_b \end{Bmatrix} = \begin{Bmatrix} Q_{nn} \\ -Q_{an} \\ -Q_{bn} \end{Bmatrix} \bar{P} + \frac{1}{2} \begin{Bmatrix} N_1 & N_2 & -N_1 & N_2 \\ -A_1 & A_2 & A_1 & -A_2 \\ -B_1 & B_2 & B_1 & -B_2 \end{Bmatrix} T$$
$$+ \begin{Bmatrix} -A_n \\ A_a \\ A_b \end{Bmatrix} q + \begin{Bmatrix} \delta \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} Q_{nn} & -Q_{na} & -Q_{nb} \\ -Q_{an} & Q_{aa} & Q_{ab} \\ -Q_{bn} & Q_{ba} & Q_{bb} \end{Bmatrix} \begin{Bmatrix} \bar{P} \\ \bar{S}_a \\ \bar{S}_b \end{Bmatrix}$$
$$+ \begin{Bmatrix} -R_n \\ R_a \\ R_b \end{Bmatrix} F \tag{60}$$

where

$$N_1 = Q_{na}M_a^2 + Q_{nb}M_b^2,\quad N_2 = Q_{na}M_a^1 + Q_{nb}M_b{}^1$$
$$A_1 = Q_{aa}M_a^2 + Q_{ab}M_b^2,\quad A_2 = Q_{aa}M_a{}^1 + Q_{ab}M_b^1$$
$$B_1 = Q_{ba}M_a^2 + Q_{bb}M_b^2,\quad B_2 = Q_{ba}M_a^1 + Q_{bb}M_b^1$$

From Eq. (49),

$$\begin{pmatrix} D_{T1}^+ - D_{T1}^- \\ D_{T2}^+ - D_{T2}^- \end{pmatrix} = \begin{pmatrix} C^1 & S^1 \\ C^2 & S^2 \end{pmatrix} \begin{pmatrix} D_a \\ D_b \end{pmatrix}$$
$$= \begin{pmatrix} C^1 & S^1 \\ C^2 & S^2 \end{pmatrix} \Bigg[ \begin{pmatrix} -Q_{an} \\ -Q_{bn} \end{pmatrix} \bar{P} + \frac{1}{2} \begin{pmatrix} -A_1 & A_2 & A_1 & -A_2 \\ -B_1 & B_2 & B_1 & -B_2 \end{pmatrix} T - \begin{pmatrix} A_a \\ A_b \end{pmatrix} q$$
$$- \begin{pmatrix} -Q_{an} & Q_{aa} & Q_{ab} \\ -Q_{bn} & Q_{ba} & Q_{bb} \end{pmatrix} \begin{bmatrix} \bar{P} \\ \bar{S}_a \\ \bar{S}_b \end{bmatrix} + \begin{pmatrix} R_a \\ R_b \end{pmatrix} F \Bigg] \tag{61}$$

where $C^1$ and $S^1$ denote diagonal matrices with $(\cos\beta_1)_j$ and $(\sin\beta_1)_j$ as their elements

Another set of equations are obtained from Eq. (57);

$$T_1^+ + T_1^- = 2m\bar{P}$$
$$T_2^+ + T_2^- = 2m\bar{P} \tag{62}$$

where $m$ is the diagonal matrix with $(\mu^*)_j$ as its elements

The global equilibrium Eqs.(35, 36) take the following form

$$H^T\bar{F} + (A_n^T\ A_a^T\ A_b^T) \begin{Bmatrix} \bar{S}_n \\ \bar{S}_a \\ \bar{S}_b \end{Bmatrix} = 0 \tag{63}$$

from which $S_a$ and $S_b$ can be eliminated using Eq. (58): Thus, Eqs. (35, 36) become

$$H^T\bar{F} - A_n^T\bar{P} + \frac{1}{2}[-E_1,\ E_2\ E_1\ -E_2]T + V^- = 0$$
$$-H^T\bar{F} + A_n^T\bar{P} - \frac{1}{2}[-E_1,\ E_2\ E_1\ -E_2]T + V^+ = 0 \tag{64}$$

where

$$E_1 = A_a^T M_a^2 + A_b^T M_b{}^2$$
$$E_2 = A_a^T M_a^1 + A_b^T M_b^1 \tag{65}$$

In summary, one obtains the following linear complementarity problem from the first equation of (60) and from (61), (62) and (64):

$$z = Mw + f$$
$$w^T z = 0,\ w \geq 0,\ z \geq 0 \tag{66}$$

where

$$z = [D_n,\ D_{T1}^+,\ D_{T2}^+,\ T_1^-,\ T_2^-,\ V^+,\ V^-]^T$$
$$w = [\bar{P},\ T_1^+,\ T_2^+,\ D_{T1}^-,\ D_{T2}^-,\ q^+,\ q^-]^T \tag{67}$$

$$M = \begin{bmatrix} Q_{nn} + N_{12}m & -N_1 & N_2 & 0 & 0 & -A_n & A_n \\ -(C^1Q_{an} + S^1Q_{bn}) \\ -[C^1A_{12} + S^1B_{12}]m & (C^1A_1 + S^1B_1) & (C^1A_2 + S^1B_2) & I & 0 & (C^1A_a + S^1A_b) & -(C^1A_a + S^1A_b) \\ -(C^2Q_{an} + S^2Q_{bn}) \\ -[C^2A_{12} + S^2B_{12}]m & (C^2A_1 + S^2B_1) & -(C^2A_2 + S^2B_2) & 0 & I & (C^2A_2 + S^2A_b) & -(C^1A_a + \dot{S}^2A_b) \\ 2m & -I & 0 & 0 & 0 & 0 & 0 \\ 2m & 0 & -I & 0 & 0 & 0 & 0 \\ A_n^T + E_{12}m & -E_1 & E_2 & 0 & 0 & 0 & 0 \\ -A_n^T - E_{12}m & E_1 & -E_2 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{68}$$

with $N_{12} = N_1 - N_2,\ A_{12} = A_1 - A_2,\ B_{12} = B_1 - B_2,\ E_{12} = E_1 - E_2$,

and

$$f = \begin{bmatrix} -Q_{nn}\bar{P} + Q_{na}\bar{S}_a + Q_{nb}\bar{S}_b - R_nF + \delta \\ (C^1Q_{an} + S^1Q_{bn})\bar{P} - (C^1Q_{aa} + S^1Q_{bn})\bar{S}_a - (C^1Q_{ab} + S^1Q_{bb})\bar{S}_b + (C^1R_a + S^1R_b)F \\ (C^2Q_{an} + S^2Q_{bn})\bar{P} - (C^2Q_{aa} + S^2Q_{bn})\bar{S}_b - (C^2Q_{ab} + S^2Q_{bb})\bar{S}_b + (C^2R_a + S^2R_b)F \\ 0 \\ 0 \\ -H^TF - H^T\bar{F} \\ H^TF + H^T\bar{F} \end{bmatrix} \tag{69}$$

When another pair of new angles, say $\alpha_3$ and $\alpha_4$, are considered, one obtains the set of equations for $D_{T3}{}^+$, $D_{T4}{}^+$, $T_3{}^-$ and $T_4{}^-$ which are the same as the equations for $D_{T1}{}^+$, $D_{T2}{}^+$, $T_1{}^-$ and $T_2{}^-$ except angles $\alpha_1$ and $\alpha_2$ replaced by $\alpha_3$ and $\alpha_4$. The two-dimensional case is recovered by setting $\alpha_1=\beta_1=0$, $\alpha_2=\beta_2=90^\circ$ and $\mu_a=\mu_b=\mu$. than $\mu_a{}^*=1$, $\mu_b{}^*=1$ and $\mu^*=\mu$.

Also in this case,

$$C^1=I,\ S^1=0,\ C^2=0,\ S^2=I.$$

Therefore, the system of LCP is given as

$$\begin{aligned} &z=Mw+f \\ &w^Tz=0,\ w\geq 0,\ z\geq 0. \end{aligned} \tag{70}$$

where

$$\begin{aligned} z&=[D_n,\ D_{T}^{+\prime},\ T_1^-,\ V^+,\ V^-]^T \\ w&=[\bar{P},\ T_1^+,\ D_{T1}^-,\ q^+,\ q^-]^T \end{aligned}$$

$$M=\begin{bmatrix} Q_{nn}+Q_{na}\mu & -Q_{na} & 0 & -A_n & A_n \\ -Q_{an}-Q_{aa}\mu & Q_{aa} & I & A_a & -A_a \\ 2\mu I & -I & 0 & 0 & 0 \\ A_n^T+A_a^T\mu & -A_a^T & 0 & 0 & 0 \\ -A_n^T-A_a^T\mu & A_a^T & 0 & 0 & 0 \end{bmatrix} \tag{71}$$

$$f=\begin{bmatrix} -Q_{nn}\bar{P}+Q_{na}\bar{S}_a-R_nF+\delta \\ Q_{an}\bar{P}-Q_{aa}\bar{S}_a+R_aF \\ -H^TF-H^T\bar{F} \\ H^TF+H^T\bar{F} \end{bmatrix} \tag{72}$$

This result is exactly the same as already obtained previously by Kwak and Lee(1988) for the two-dimensional case.

A couple of observations are in order. When $\mu_a=\mu_b=0$, than $m=0$. In this frictionless case, the martix $M$ for both three-dimensional and two-dimensional case takes the following form,

$$\begin{bmatrix} Q & A \\ -A^T & 0 \end{bmatrix} \tag{73}$$

which corresponds to a strictly convex quadratic programming problem, when $Q$ is a symmetric positive definite matrix, as is the case for stable equilibrium problems. However, when $m$ is not zero, it is true that no corresponding minimization formulation is available and no unique solution can be expected. This question will be examined further elsewhere. The same statement holds true for two-dimensional case as can be seen immediately from Eq. (71).

# 6. REDUCTION TO SIMPLER CASES

## 6.1 Two-Dimensional Frictional Contact

For two dimensional case, the Coulomb law says that $|S_T|\leq\mu P$. This can be equivalently stated as,

$$\begin{aligned} S_T+T^+&=\mu P \\ -S_T+T^-&=\mu P \end{aligned} \tag{74}$$

where slack varibles are introduced as before. The relative slip $D_T$ is accordingly expressed as a difference of two nonnegative values;

$$D_T=D_T{}^+-D^{T-} \tag{75}$$

where

$$D_T=D_T(S_T,\ P\,;F) \tag{76}$$

Then the following complementarities hold,

$$D_T{}^+T^+=0 \text{ and } D_T{}^-T^-=0. \tag{77}$$

By eliminating $S_T$ from Eq. (74),

$$T^++T^-=2\mu P \tag{78}$$

Also, since $S_T=(T^+-T^-)/2$, $S_T$ can be eliminated from all other expressions such as $D_n$ and $D_T$. Threfore, Eqs.(35, 36, 43, 75, 78) constitute the complementarity problem sought for with complementarity relations (33), (37) and (77). The resulting system is linear and found to be the same as system(70).

## 6.2 Reducton to Frictioless Contact

If there is no friction, than the friction condition described in Section 2 is not necessary and all tangential contact force is zero. The basic equations are simplified. Eq.(29) now becomes,

$$u_c^k=u_c^k({}^{t+\Delta t}S_n^k\,;{}^{t+\Delta t}F^k),\ \ k=1,2. \tag{79}$$

From Eqs. (35, 36)

$$\begin{aligned} E({}^tS_n+S_n\,;{}^tF+F)-V^-&=0 \\ E({}^tS_n+S_n\,;{}^tF+F)-V^+&=0 \end{aligned} \tag{80}$$

Using Eq. (79), Eq. (31) becomes

$$D_n({}^ta_i^1)=G^1({}^tS_n+S_n,\ q^+,\ q^-\,;{}^ta_i^1,\ {}^ta_i^2) \tag{81}$$

The complementarity conditions are, with $P=-S_n$,

$$({}^tP+P)D_n=0 \tag{82}$$

and

$$V^+q^+=0 \text{ and } V^-q^-=0 \tag{83}$$

with all variables nonnegative as in (38).

The system (80) to (83) constitute a complementarity problem sought for. For the particular case of linear elastic, small displacement, frictionless contact, it is a linear complementarity problem and is equivalent to the well-known quadratic programming approach derived from the principle of minimum potential energy under kinematic constraints. The linear complementarity problem corresponds to necessary conditions of this quadratic programming problem.

# 7. SUMMARY AND CONCLUSIONS

It has been shown that a three dimensional frictional contact problem can be described by a complementarity principle. The fact that it has inherent nonlinearity in the expression even for the case of linearly elastic materials contrasts itself with the linear complementarity problem formulation possible for two dimensional frictional problem.

The formulation is very general without any specific restrictions such as the assumptions on normal contact force or on contact area usually made in the literature. It is applicable to nonliner material and geometric problems with orthortropic law of friction. A linearization suitable for a numerical analysis has been possible and implemented as an explicit LCP form by introduction of a polyhedral law of friction approximating the original elliptic law. The complementarity problem previously introduced by the author for two dimensional contact has been shown to be a special case of this linearized three dimensional formulation.

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